Adaptive Tangential Relaxation and Surface Fitting for High-Order Mesh Optimization

29th International Meshing Roundtable
(virtual conference)

CASC
Center for Applied Scientific Computing

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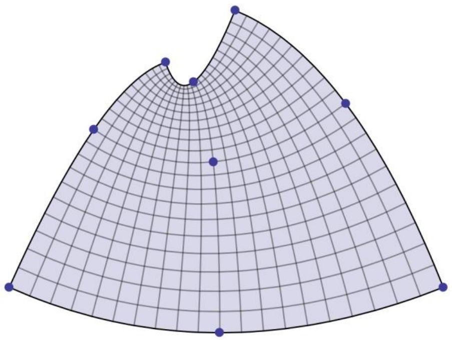
Goal: derive mesh optimization methods based on finite element operations

- The mesh positions are represented as a FE function.

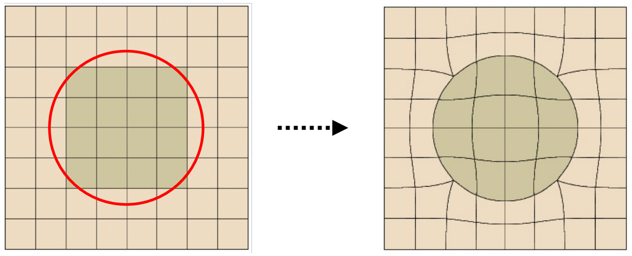
$$\boldsymbol{x} = (\boldsymbol{x}_1 \dots \boldsymbol{x}_N)^T, \quad x_q(\bar{x}_q) = \sum_{i=1}^{N} \boldsymbol{x}_i \bar{w}_i(\bar{x}_q)$$

$$A_q(x) = \frac{\partial x_q}{\partial \bar{x}_q} = \sum_{i=1}^{N} \boldsymbol{x}_i [\nabla \bar{w}_i(\bar{x}_q)]^T$$



Example of a single Q_2 element

- Node movement with fixed topology.
  - Generality w.r.t. dimension and element type.
  - Avoid geometric operations.
  - Computational performance.

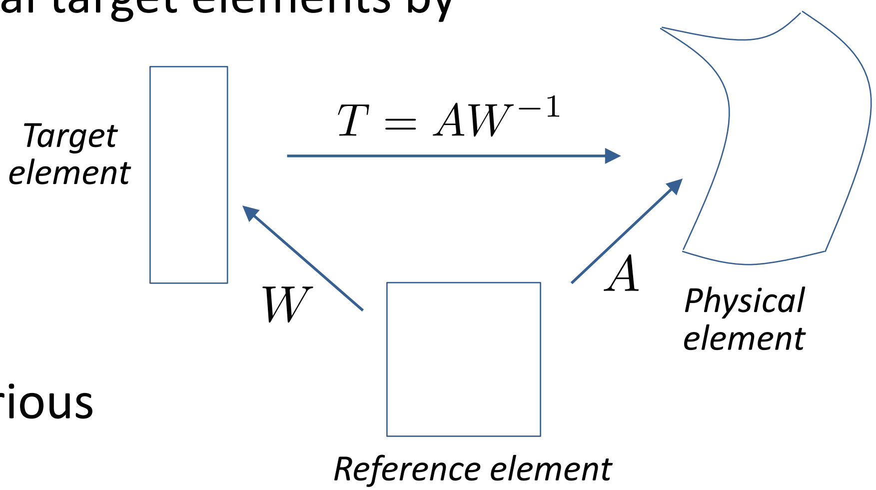


Example of fitting through node movement

- Focus of this talk: extend the approach to tangential optimization and fitting.
  - Implicitly defined surfaces (level set functions).

Approach overview: Target-Matrix Optimization Paradigm (TMOP) and variational minimization

- Target construction: the user defines ideal target elements by specifying the target Jacobians W.



- The Jacobian T is used to define the local mesh quality measure $\mu(T)$.

- Combinations of W and $\mu(T)$ control various properties of the physical elements.
  $W =$ [volume] [orientation] [skew] [aspect ratio].

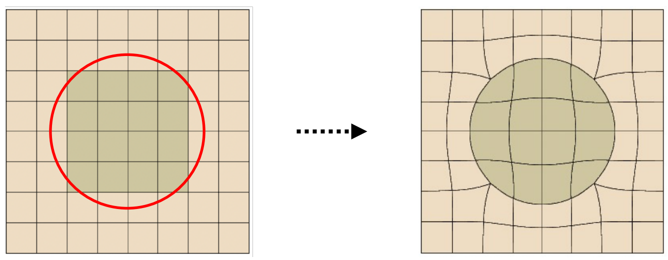
- Variational minimization over the target elements (solving $\partial F(x) \,/\, \partial \boldsymbol{x} = 0$):

$$F(x) := \sum_{E \in \mathcal{M}} \int_{E_t} \mu(T(x_t)) dx_t = \sum_{E \in \mathcal{M}} \sum_{x_q \in E_t} w_q \det(W(\bar{x}_q)) \, \mu(T(x_q))$$

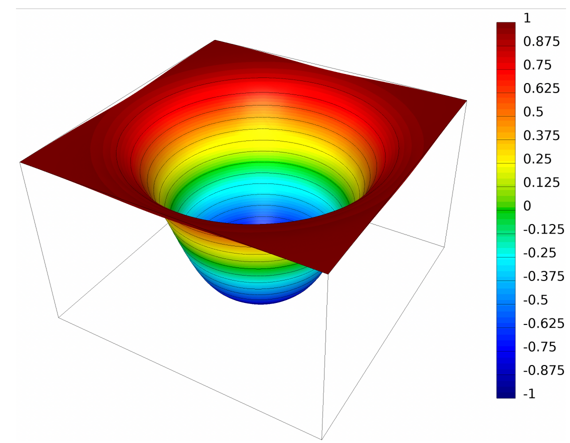
Dobrev, Knupp, Kolev, Mittal, Tomov, "The Target-Matrix Optimization Paradigm for high-order meshes", SISC, 2019
Knupp, "Metric Type in the Target-matrix Mesh Optimization Paradigm" LLNL-TR-817490, 2020.

Fitting and tangential relaxation are enforced weakly by a variational penalty term

- The surface of interest is given as a discrete level set (no analytic parametrization).



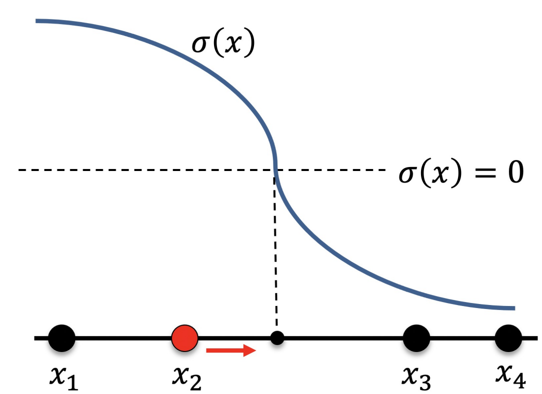
Example of 2D interface fitting



- Penalty formulation (quality / fitting tradeoff).
  - All mesh nodes move simultaneuously.
  - One approach for fitting / tangential relaxation.

$$F(x) = F_\mu + w_\sigma \int_\Omega \bar{\sigma}(x)^2$$

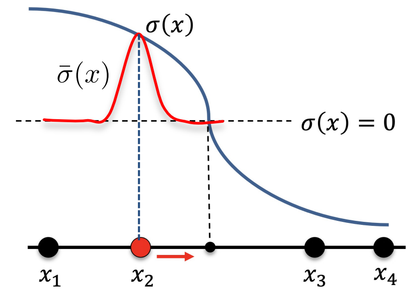
- The restricted level set function $\bar{\sigma}$ penalizes the deviation from the zero level set.
  - Marking is not a trivial procedure.



The extra term affects only the position of the red DOF

The penalty term is put in a form that can be easily differentiated

- We restrict σ to the set of marked nodes: $$\bar{\sigma}_i = \begin{cases} \sigma_i & \text{if } i \in \mathcal{S}, \\ 0 & \text{otherwise.} \end{cases}$$
- Goal: move the mesh so that $\sigma_i \equiv \sigma(x_i) = 0$.
  - Interpolatory finite element basis functions.
  - The nodes of x and σ must coincide.



- The derivative computation must see the term $\partial\sigma_i \backslash \partial x \neq 0$.

$$\bar{\sigma}(\boldsymbol{x}) = \sum_{i \in \mathcal{S}} \sigma_i \phi_i(\boldsymbol{x}) = \sum_{s \in \mathcal{S}} \sum_{k=1}^{N_x} \sigma_k \phi_k(\boldsymbol{x_i}) \phi_i(\boldsymbol{x}).$$

$$\phi_k(\boldsymbol{x_s}) = \delta_{ks}$$

Derivatives of this term represent the changes of σ as the mesh evolves

- Field remap – required as σ is a discrete FE function.

Dobrev, Knupp, Kolev, Tomov, "Towards simulation-driven optimization of high-order meshes by TMOP", IMR, 2019.

Derivatives of the penalization term are calculated through standard FE operations

$$\frac{\partial F_\sigma}{\partial x_{a,i}} = \frac{2\omega_\sigma}{c_\sigma}\int_{E_t}\bar{\sigma}(\boldsymbol{x})\frac{\partial\bar{\sigma}(\boldsymbol{x})}{\partial x_a}\frac{\partial x_a}{\partial x_{a,i}}$$

$$= \frac{2\omega_\sigma}{c_\sigma}\int_{E_t}\bar{\sigma}(\boldsymbol{x})\sum_{s\in\mathcal{S}}\sum_k \sigma_k\left(\frac{\partial\phi_k(\boldsymbol{x_s})}{\partial x_a}\phi_s(\boldsymbol{x})+\phi_k(\boldsymbol{x_s})\frac{\partial\phi_s(\boldsymbol{x})}{\partial x_a}\right)w_i(\bar{\boldsymbol{x}}),$$

$$\frac{\partial^2 F_\sigma}{\partial x_{b,j}\partial x_{a,i}} = \frac{2\omega_\sigma}{c_\sigma}\int_{E_t}\left(\frac{\partial\bar{\sigma}(\boldsymbol{x})}{\partial x_b}\frac{\partial\bar{\sigma}(\boldsymbol{x})}{\partial x_a}+\bar{\sigma}(\boldsymbol{x})\frac{\partial^2\bar{\sigma}(\boldsymbol{x})}{\partial x_b\partial x_a}\right)\frac{\partial x_a}{\partial x_{a,i}}\frac{\partial x_b}{\partial x_{b,i}}$$

$$= \frac{2\omega_\sigma}{c_\sigma}\int_{E_t}\left(\mathcal{D}_a\mathcal{D}_b+\bar{\sigma}(\boldsymbol{x})\mathcal{D}^2\right)w_i(\bar{\boldsymbol{x}})w_j(\bar{\boldsymbol{x}}),$$

where

$$\mathcal{D}_* = \sum_{s\in\mathcal{S}}\sum_k \sigma_k\left(\frac{\partial\phi_k(\boldsymbol{x_s})}{\partial x_*}\phi_s(\boldsymbol{x})+\phi_k(x_s)\frac{\partial\phi_s(\boldsymbol{x})}{\partial x_*}\right),$$

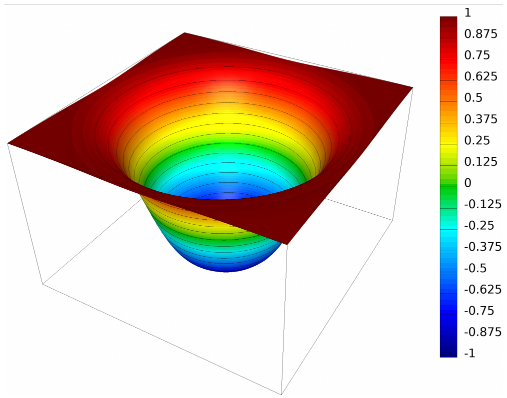
$$\mathcal{D}^2 = \sum_{s\in\mathcal{S}}\sum_k \sigma_k\left(\frac{\partial\phi_k(\boldsymbol{x_s})}{\partial x_a}\frac{\partial\phi_s(\boldsymbol{x})}{\partial x_b}+\frac{\partial^2\phi_k(\boldsymbol{x_s})}{\partial x_b\partial x_a}\phi_s(\boldsymbol{x})+\frac{\partial\phi_k(\boldsymbol{x_s})}{\partial x_b}\frac{\partial\phi_s(\boldsymbol{x})}{\partial x_a}+\phi_k(\boldsymbol{x_s})\frac{\partial^2\phi_s(\boldsymbol{x})}{\partial x_b\partial x_a}\right),$$

$$a,b=1\ldots d,\quad i,j=1\ldots N_x.$$

Knupp, Kolev, Mittal, Tomov, "Adaptive surface fitting and tangential relaxation for high-order mesh optimization", IMR, 2021.

The method behaves as expected on academic problems with smooth interfaces

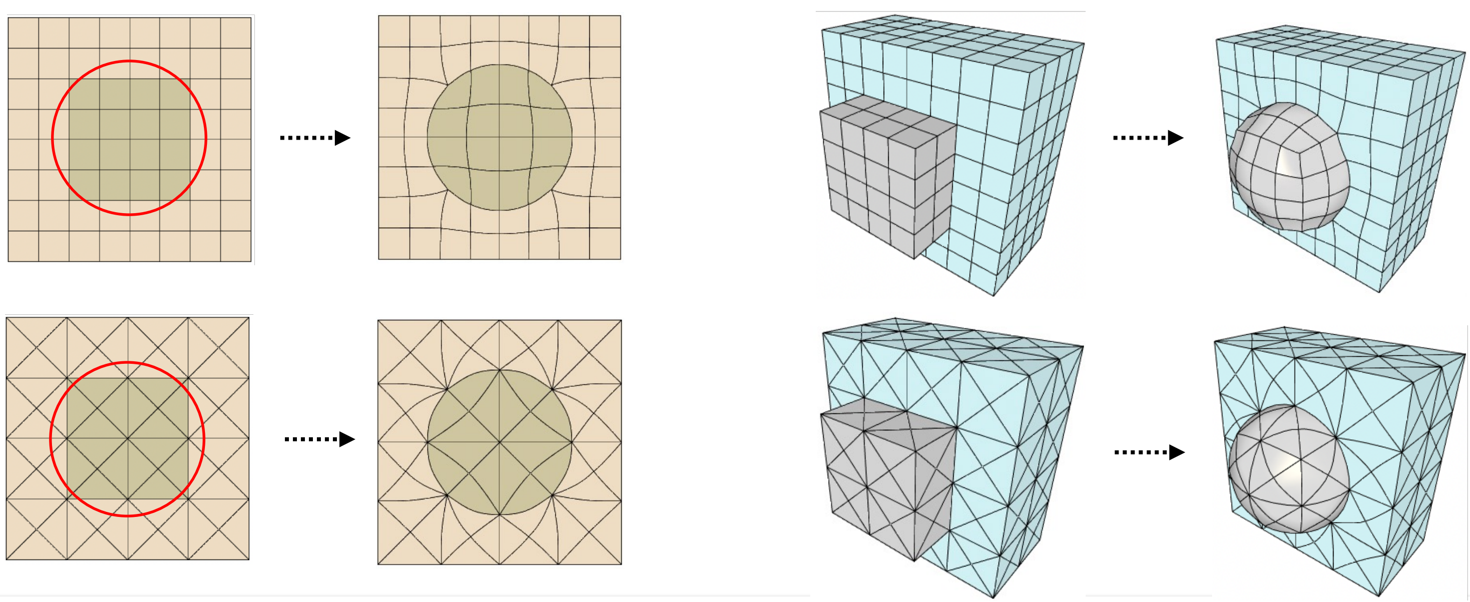
- In all cases σ is a discrete FE function.
  - Ball at the center of the domain.



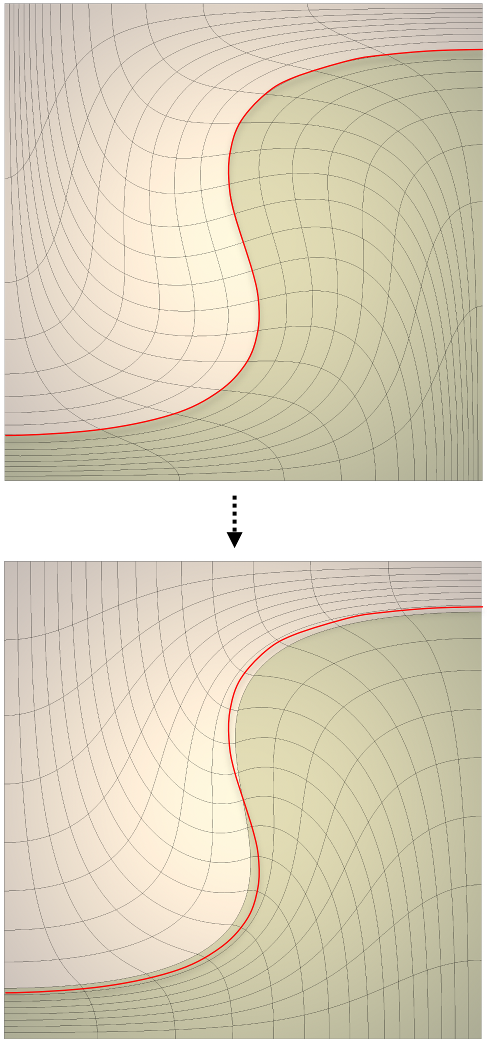
2D metric: $$\mu_{80} = (1-\gamma)\left(0.5\frac{|T|^2}{\tau} - 1\right) + \gamma\left(0.5\left(\tau - \frac{1}{\tau}\right)^2\right)$$

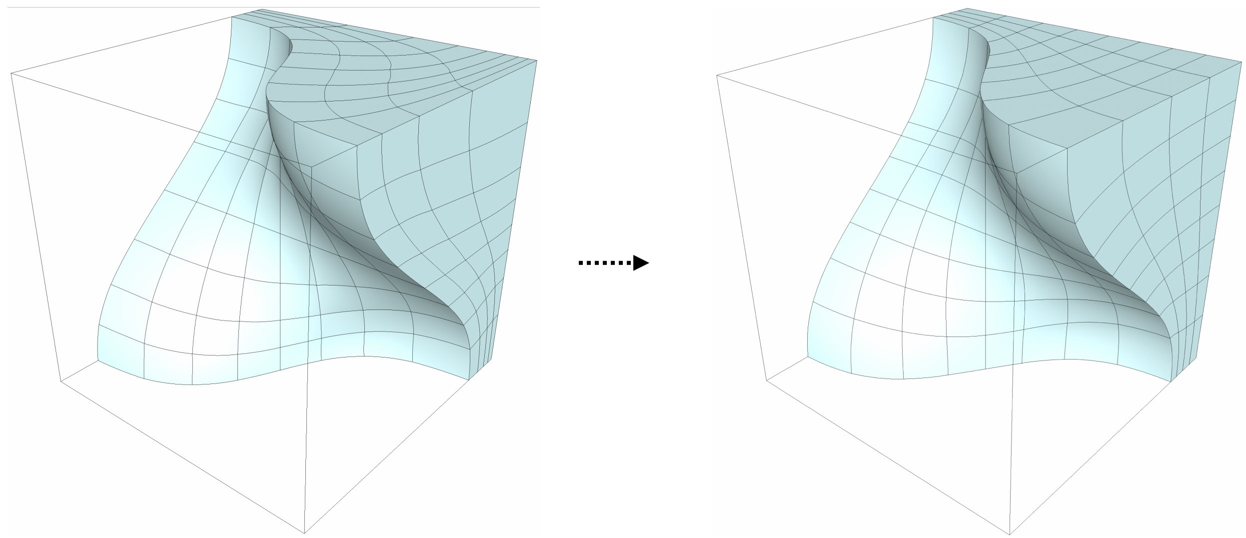
3D metric: $$\mu_{333} = (1-\gamma)\left(\frac{|T|^2|T^{-1}|^2}{9} - 1\right) + \gamma\left(0.5\left(\tau + \frac{1}{\tau}\right) - 1\right)$$

P. Knupp, "Metric Type in the Target-matrix Mesh Optimization Paradigm" LLNL-TR-817490, 2020.



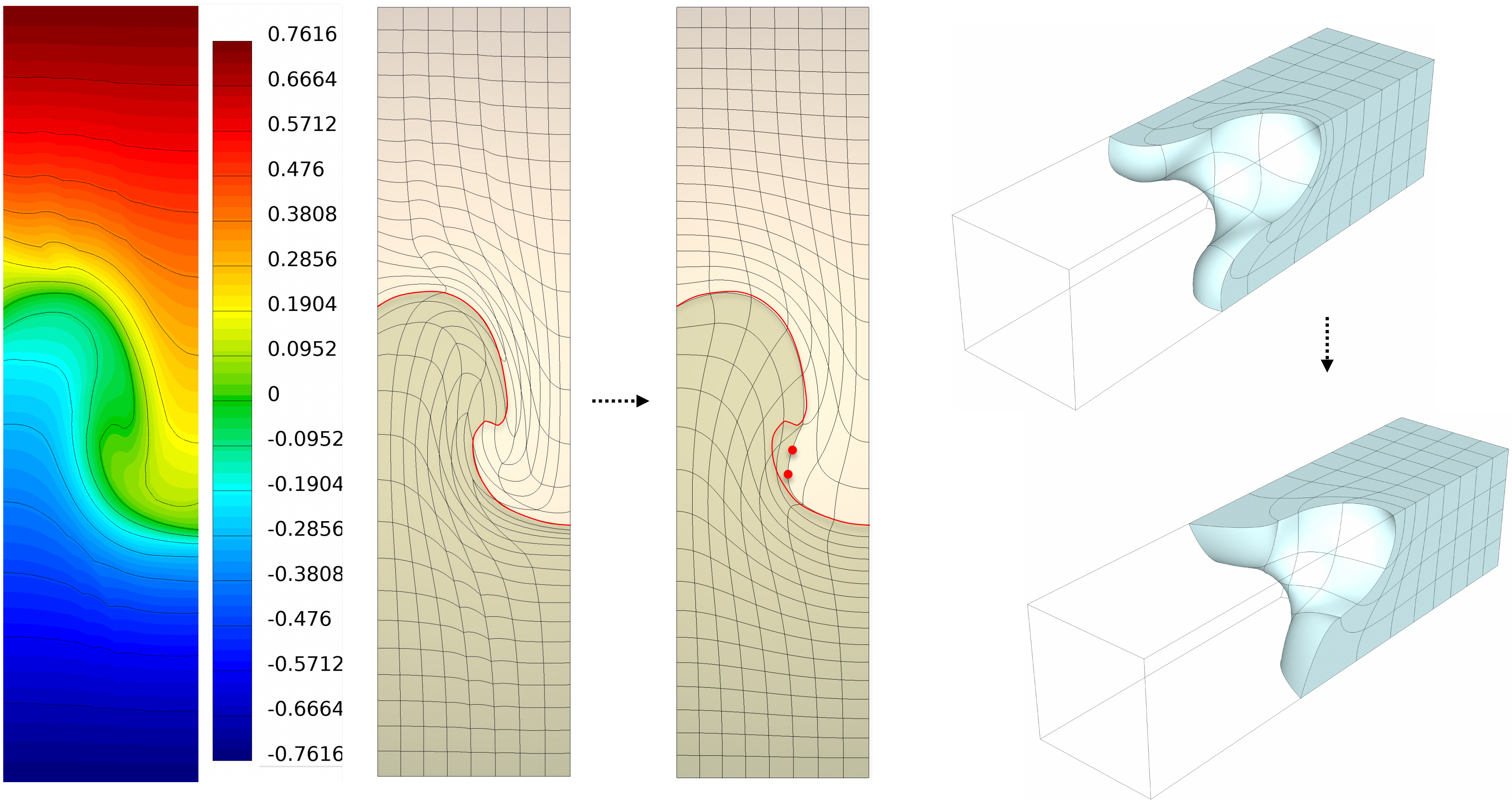
The method behaves as expected on academic problems with smooth interfaces





Approach	F decrease	$\mathcal{E}_{avg}$	$\mathcal{E}_{max}$
Fixed interface	34.4%	0	0
$w_\sigma = 250$	51.4%	8.2e-2	1.3e-1
$w_\sigma = 1000$	42.6%	3.6e-2	6.4e-2

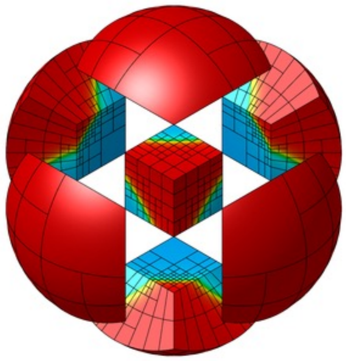
For non-smooth practical cases, further research and methods are required



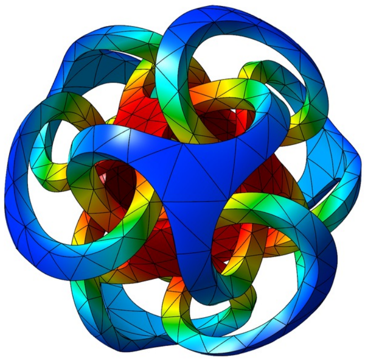
MFEM open source implementation

- All presented methods are (or will be) available in MFEM.

- MFEM contains **12** 2D metrics, **7** 3D metrics, all metric derivatives, **6** target construction methods.



mfem.org



glvis.org

- User interface provided by the ***mesh_optimizer*** and ***pmesh_optimizer*** miniapps.
  - Choice of target construction / quality metric / adaptivity fields / parameters.
  - Visualization through GLVis.

Dobrev, Knupp, Kolev, Mittal, Tomov, “The Target-Matrix Optimization Paradigm for high-order meshes”, SIAM J. Sci. Comp., 2019.
Dobrev, Knupp, Kolev, Mittal, Rieben, Tomov, “Simulation-driven optimization of high-order meshes in ALE hydrodynamics”, Comput. Fluids, 2020.
Knupp, Kolev, Mittal, Tomov, “Adaptive surface fitting and tangential relaxation for high-order mesh optimization”, IMR, 2021.

Summary and future work

- Surface fitting and tangential relaxation through an adaptive FE formulation.
  - Discrete representation of the surface; no analytic parametrization.
  - Exploratory attempt to use FE-only operations.
- Weak enforcement through a variational penalty term.
  - Simultaneous optimization of surface and non-surface nodes.
  - Applicable to both fitting and tangential relaxation.
- Derivatives are computed through standard FE operations.
- Behaves as expected on smooth surfaces.
- Future work – robust behavior for practical non-smooth problems.

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